Decay Constant of Pseudoscalar Meson in the Heavy Mass Limit

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Abstract

The leptonic decay constant of the pseudoscalar mesons a calculated by use of the relativistic constituent quark model constructed on the point form of Poincare-covariant quantum mechanics. We discuss the role relativistic corrections for decay constants of pseudoscalar mesons with heavy quarks. We consider the heavy mass limit of decay constant for two-particle system with equal masses.

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1 Introduction

According to the structure of the current interaction, electroweak decays of hadrons can be divided into following classes: leptonic decays,in which the quarks of the decaying hadron annihilate each other and only leptons appear in the final state; semileptonic decays, in which both leptons and hadrons appear in the final state; photons decays,in which the final state consists of photons only; radiative transitions between hadrons, in which hadrons and photon are caused by hadron decays. non-leptonic decays,in which the final state consists of hadrons only. Over the last decade, a lot of information on hadron decays has been collected in experiments at e^+e^- and hadron colliders. This has led to a rather detailed knowledge of the flavour sector of the Standard Model and many of the parameters associated with it. In this work we investigate the decay constant of the mesons with spinor quarks in the heavy mass limit.

2 $q\overline{q}$ bound state in the point form of RQM

There are three forms of the dynamics in the relativistic quantum mechanics (RQM), called instant ,point,light-front forms [1]. In this work we use point form of the RQM [2]. The description in the point form implies that the generators of the Poincare group $\hat{M}^{\mu\nu}$ are the same as for noninteracting particles and bound systems. Interaction terms can be present only in the four-momentum operators $\hat{P}_\mu$, but the four-velocities of bound and free-particle systems are equal.

The momenta $\vec{p}_1, \vec{p}_2$ of the quarks with the masses m_1 and m_2 of relativistic system can be transformed to the total $\vec{P}$ and relative momenta $\vec{k}$ to facilitate the separation of the center mass motion:

$$\vec{P}_{12} = \vec{p}_1 + \vec{p}_2,$$

$$\overrightarrow{k} = \overrightarrow{p_1} + \frac{\overrightarrow{P_{12}}}{M_0}\left(\frac{\left(\overrightarrow{P_{12}}\overrightarrow{p_1}\right)}{\omega_{M_0}\left(\overrightarrow{P_{12}}\right) + M_0} + \omega_{m_1}\left(\overrightarrow{p_1}\right)\right), \quad (1)$$

where $M_0 = \omega_{m_1}\left(\overrightarrow{k}\right) + \omega_{m_2}\left(\overrightarrow{k}\right), \omega_{m_1}\left(\overrightarrow{p_1}\right) = \sqrt{\overrightarrow{p_1}^2 + m_1^2}$.

The solution of the eigenvalue problem will lead to eigenfunction of the form

$$_0\left\langle \vec{V}_{12}\mu, [J\,k], (ls)\left|\vec{V}\mu, [J\,M]\right.\right\rangle =$$

$$= \delta_{JJ'}\delta_{\mu\mu'}\delta(\vec{V} - \vec{V}_{12})\Psi^{J\mu}(k\,l\,s\,; M) \quad (2)$$

with the velocities of bound system $\vec{V} = \vec{P}/M$ and noninteracting system $\vec{V}_{12} = \vec{P}_{12}/M_0$. The function $\Psi^{J\mu}(k\,l\,s\,; M)$ satisfies in the point form a following equation [4]:

$$\sum_{l's'}\int_0^\infty < k\,l\,s \parallel W^J \parallel k'\,l'\,s' > \Psi^J(k'\,l'\,s';\, M)k'^2 dk' +$$

$$+ k^2\Psi^J(k\,l\,s; M) = \eta\Psi^J(k\,l\,s; M) \quad (3)$$

with reduced matrix element of operator $\hat{W}$.

In the point form the meson state is defined by as state of on-shell quark and antiquark with the wave function $\Psi^{J\mu}(k\,l\,s\,; M)$

$$\left|\overrightarrow{P}\mu\;[JM]\right\rangle = \sqrt{\frac{M}{\omega_M\left(\overrightarrow{P}\right)}} *$$

$$* \sum_{ls\lambda_1\lambda_2}\int d^3k\sqrt{\frac{\omega_{m_1}\left(\overrightarrow{p_1}\right)\omega_{m_2}\left(\overrightarrow{p_2}\right)}{\omega_{m_1}\left(\overrightarrow{k}\right)\omega_{m_2}\left(\overrightarrow{k}\right)}}\Psi^{J\mu}(kls; M)$$

$$\sum_{m\lambda}\sum_{\nu_1\nu_2}\langle s_1\nu_1, s_2\nu_2|\, s\lambda\rangle\,\langle lm, s\lambda|\, J\mu\rangle\, Y_{lm}(\theta,\phi)$$

$$D^{1/2}_{\lambda_1\nu_1}\left(\overrightarrow{n}\left(p_1, P\right)\right) D^{1/2}_{\lambda_2\nu_2}\left(\overrightarrow{n}\left(p_2, P\right)\right)|p_1\lambda_1\rangle\,|p_2\lambda_2\rangle \quad (4)$$

where $\langle s_1\nu_1, s_2\nu_2|\, s\lambda\rangle$, $\langle lm, s\lambda|\, J\mu\rangle$ are Clebsh-Gordan coefficients of $SU(2)$-group, $Y_{lm}(\theta,\phi)$ - spherical harmonic with spherical angle of $\vec{k}$. Also, in Eq.(4) $D^{1/2}\left(\overrightarrow{n}\right) = 1 - i\left(\overrightarrow{n}\,\overrightarrow{\sigma}\right)/\sqrt{1+\overrightarrow{n}^2}$is D -function of Wigner rotation, which determined by vector-parameter $\overrightarrow{n}(p_1, p_2) = \overrightarrow{u_1} \times \overrightarrow{u_2}/(1 - (\overrightarrow{u_1}\overrightarrow{u_2}))$ with $\overrightarrow{u} = \overrightarrow{p}/\left(\omega_m\left(\overrightarrow{p}\right) + m\right)$.

3 Leptonic decay constant

The leptonic decay constant for pseudoscalar meson is defined by

$$\left\langle 0 \left| \hat{J}^{\mu}(0) \right| \overrightarrow{P}, M \right\rangle = i\,(1/2\pi)^{3/2} \frac{1}{\sqrt{2\omega_M\left(\overrightarrow{P}\right)}} P^{\mu} f_p, \tag{5}$$

where $\hat{J}^{\mu}(0)$ is the operator axial-vector part of the charged weak current. Using Eq.(4) and Eq.(5) we found in the point form dynamics, that [5]

$$f_p = \frac{N_c}{\pi\sqrt{2}} \int_0^{\infty} dk\, k^2 \sqrt{\frac{M_0^2 - (m_1 - m_2)^2}{\omega_{m_1}\left(\overrightarrow{k}\right)\omega_{m_2}\left(\overrightarrow{k}\right)}} *$$

$$* \frac{(m_1 + m_2)}{M_0^{3/2}} \Psi(k, M), \tag{6}$$

where N_c-number of colors, m_1 and m_2 are the respective masses of the two quarks. The wave function for pseudoscalar meson have the normalization

$$\int_0^{\infty} dk\, k^2\, N_c \left|\Psi(k, M)\right|^2 = 1.$$

When $m_1 = m_2 = m_Q$, the leptonic decay constant is defined by

$$f_p = \frac{2N_c m_Q}{\pi} \int_0^{\infty} \frac{dk\, k^2 \Psi(k, M)}{\omega_{m_Q}^{3/2}\left(\overrightarrow{k}\right)}. \tag{7}$$

The equation for the bound $q\bar{q}$ states (3) in the RQM is relativistic equation with effective potential W . However, it is hard problem to obtain wave function $\Psi(k, M)$ as solution of this equation. Therefore, we use simple model wave function depending on length scale parameter $1/\beta$:

$$\Psi(k\,, M) \equiv \Psi(k\,, \beta) = 2/(\sqrt{N_c}\beta^{3/2}\pi^{1/4}) exp(-\frac{k^2}{2\beta^2}). \tag{8}$$

Using the equations (7) and (8), one can see that

$$\begin{aligned} f_{\pi} \quad = \quad & \frac{\sqrt{N_c}\beta}{\pi^{5/4}\Gamma\left(-\frac{1}{4}\right)W} \\ & (2^{3/4}\Gamma\left(-\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right){}_1F_1\left(\frac{3}{4};\frac{1}{4};\frac{1}{2W^2}\right) - \\ & \frac{2\sqrt{\pi}}{W^{3/2}}\Gamma\left(-\frac{3}{4}\right){}_1F_1\left(\frac{3}{2};\frac{7}{4};\frac{1}{2W^2}\right)), \end{aligned} \tag{9}$$

with $W = \beta/m_Q$, hypergeometric function ${}_1F_1(a;b;z)$ and $\Gamma(z)$ -Gamma function.

We now consider the heavy mass limit of (6). This limit is defined as m_1, $m_2 \longrightarrow \infty$ with $V = P/M$ fixed. The starting point in the construction of the effective theory with the heavy quarks (HQET) is the observation that a heavy quark bound inside a hadron moves more or less with the hadron's velocity V. Its momentum can be written as

$$p_Q = m_Q V + \widetilde{k}, \tag{10}$$

where the components of the so-called residual momentum $\widetilde{k}$ are much smaller than m_Q. Interactions of the heavy quark with light degrees of freedom change the residual momentum by an amount of order $\widetilde{k} \sim \Lambda_{QCD} \simeq 1/R_{Hadron}$, but the corresponding changes in the heavy-quark velocity vanish as $\Lambda_{QCD}/m_Q \longrightarrow 0$. In the system of the center mass we are obtained, that relative momentum $\overrightarrow{k}$ (1) and residual momentum $\overrightarrow{\widetilde{k}}$ are equal and therefore, the heavy mass limit in the point form is given by

$$\left|\overrightarrow{k}\right| \leq \Lambda_{QCD} \ll m_Q. \tag{11}$$

The nonrelativistic variant furnishes the following relationship for leptonic decay constant (6):

$$\begin{aligned} f_{nonrel} \quad = \quad & \frac{2N_c m_Q}{\pi} \int_0^{\Lambda_{QCD}} dk\, k^2 \\ & (1 - \frac{3k^2}{4m_Q^2} + \frac{21k^4}{32m_Q^4})\Psi_{nonrel}(k, M), \end{aligned} \tag{12}$$

where $\Psi_{nonrel}(k, M)$ have the normalization

$$\int_0^{\Lambda_{QCD}} dk\, k^2\, N_c \left|\Psi_{nonrel}(k, M)\right|^2 = 1.$$

The parameter β can be estimate from mean square radius (MSR) $\langle r^2 \rangle_{nonrel} = 1/\Lambda_{QCD}^2$ of the meson with the heavy quarks. In the nonrelativistic approximation the MSR is $\langle r^2 \rangle_{nonrel} = 3/8\beta^2$ and we obtain the relationship between β and Λ_{QCD}:

$$\Lambda_{QCD} = \sqrt{\frac{8}{3}}\beta. \tag{13}$$

The nonrelativistic wave function can be choose the form:

$$\Psi_{nonrel}(k, M) \sim \Psi(k, M) \sim exp(-\frac{k^2}{2\beta^2}), \tag{14}$$

since the model wave function (8) has not the small parameter Λ_{QCD}/m_Q (or $W = \beta/m_Q$).Using (12),(13) and (14) we obtain the following result for f_{nonrel}:

$$f_{nonrel} \approx \sqrt{N_c}\beta\sqrt{W}(0.72 - 0.73W^2 + 1.09W^4). \quad (15)$$

Let us discuss in brief the role of relativistic corrections in leptonic decay constant of pseudoscalar meson with heavy quarks. This effect can be extracted easily. Using asymptotic limit for Kummer's function ($1/W \longrightarrow \infty$)we found, that decay constant (9) can be written as

$$\begin{aligned} f_p &\approx \frac{\sqrt{N_c}\beta\sqrt{W}}{\pi^{3/4}16\sqrt{2}}\left(32 - 72W^2 + 315W^4\right) \\ &\approx \sqrt{N_c}\beta\sqrt{W}(0.60 - 1.35W^2 + 5.90W^4). \quad (16) \end{aligned}$$

Comparison of series (16) and (15), that the factors at addenda of these series differ, especially second and third term of a series. Just, these addenda also give corrections to the effective theory of heavy quarks. If we let's assume, that the parameter $\Lambda_{QCD} = a\beta$ with $a = \sqrt{2}$, the first terms of series practically coincides for two variants

$$f_{nonrel} \approx \sqrt{N_c}\beta\sqrt{W}(0.60 - 0.47W^2 + 0.55W^4), \quad (17)$$

but the second and third terms nevertheless essentially differ. Practically we compare two approaches of an evaluation of relativistic corrections for the effective theory of heavy quarks: the first approach follows from an exact solution of a problem with a consequent passage to the limit of heavy quarks; the second approach is based on an approximate solution of a problem; In the second approach, and such approach, as we see requires cutting relative momentum by magnitude Λ_{QCD} in a quark model of a meson, relativistic corrections has the smaller value just because of cutting. Such divergence can be reduced by introduction of a parameter μ , which $\gg \Lambda_{QCD}$. However, it can be defined a value only using exact calculation. Therefore use of exact expressions for observable magnitudies is represented preferable to us, as, the numerical integration both approximate relations, and exact expressions has an identical order of complexity.

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